

RENORMALIZATION GROUP IN THE STATISTICAL THEORY OF TURBULENCE: TWO-LOOP APPROXIMATION

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The field theoretic renormalization group is applied to the stochastic Navier–Stokes equation that describes fully developed fluid turbulence. The complete two-loop calculation of the renormalization constant, the beta function and the fixed point is performed. The ultraviolet correction exponent, the Kolmogorov constant and the inertial-range skewness factor are derived to second order of the ε expansion. Submitted to **Acta Physica Slovaca**.

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I. INTRODUCTION

One of the oldest open problems in theoretical physics is that of describing fully developed turbulence on the basis of a microscopic model. The latter is usually taken to be the stochastic Navier–Stokes (NS) equation subject to an external random force which mimics the energy input by the large-scale modes; see, e.g., [1,2]. The aim of the theory is to verify the basic principles of the celebrated Kolmogorov–Obukhov phenomenological theory, study deviations from this theory, determine the dependence of various correlation functions on the times, distances, external (integral) and internal (viscous) turbulence scales, and derive the corresponding scaling dimensions. Most results of this kind were obtained within the framework of numerous semiphenomenological models which cannot be considered to be the basis for construction of a regular expansion in certain small (at least formal) parameter [1,2].

An important exception is provided by the renormalization group (RG) method that was earlier successfully applied in the theory of critical behaviour to explain the origin of critical scaling and to calculate universal quantities (critical dimensions and scaling functions) in the form of the ε expansions [3].

The RG approach to the stochastic NS equation, pioneered in [4–7], allows one to prove the existence of the infrared (IR) scale invariance with exactly known “Kolmogorov” dimensions and the independence of the correlation functions of the viscous scale (the second Kolmogorov hypothesis), and calculate a number of representative constants in a reasonable agreement with experiment. Detailed review of the RG theory of turbulence and more references can be found in [8,9].

In contrast to the standard ϕ^4 model of critical behaviour [3], where the critical exponents are known up to the order ε^5 (five-loop approximation), all the calculations in the RG approach to the stochastic NS equation have been confined with the simplest one-loop approximation. The reason for this distinction is twofold. First, the multiloop calculations for this dynamical model are rather involved: one can say that the two-loop calculation for the stochastic NS equation is as cumbersome as the four-loop calculation for the conventional ϕ^4 model. Second, the critical dimensions for the most important physical quantities (velocity and its powers, frequency, energy dissipation rate and so on) are given by the one-loop approximation exactly (the corresponding ε series terminate at first-order terms) and the higher-order calculations for them are not needed.

However, the ε series for other important quantities do not terminate and the calculation of the higher-order terms for them is of great interest. In this paper, we present the results of the two-loop calculation for a number of such quantities: the ultraviolet (UV) correction exponent ω , the Kolmogorov constant C_K and the inertial-range skewness factor \mathcal{S} .

II. THE MODEL, FIELD THEORETIC FORMULATION AND RENORMALIZATION

Detailed exposition of the RG theory of turbulence and the bibliography can be found in [8,9]; below we restrict ourselves to only the necessary information.

As the microscopic model of the fully developed, homogeneous, isotropic turbulence of an incompressible viscous fluid one usually takes the stochastic NS equation with a random driving force

$$\nabla_t \varphi_i = \nu_0 \partial^2 \varphi_i - \partial_i P + F_i, \quad \nabla_t \equiv \partial_t + (\varphi \partial). \quad (2.1)$$

Here φ_i is the transverse (due to the incompressibility) three-dimensional vector velocity field, P and F_i are the pressure and the transverse random force per unit mass (all these quantities depend on $x \equiv \{t, \mathbf{x}\}$), ν_0 is the kinematical

viscosity coefficient, ∂^2 is the Laplace operator and ∇_t is the Lagrangian derivative. The problem (2.1) is studied on the entire t axis and is augmented by the retardation condition and the condition that φ_i vanishes for $t \rightarrow -\infty$. We assume for F a Gaussian distribution with zero mean and correlator

$$\langle F_i(x) F_j(x') \rangle = \delta(t - t') (2\pi)^{-3} \int d\mathbf{k} P_{ij}(\mathbf{k}) d_F(k) \exp[i\mathbf{k}(\mathbf{x} - \mathbf{x}')], \quad (2.2)$$

where $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$ is the transverse projector and $d_F(k)$ is some function of $k \equiv |\mathbf{k}|$ and model parameters.

The stochastic problem (2.1), (2.2) is equivalent to the field theoretic model of the doubled set of fields $\Phi \equiv \{\varphi, \varphi'\}$ with action functional

$$S(\Phi) = \varphi' D_F \varphi' / 2 + \varphi' [-\partial_t \varphi + \nu_0 \partial^2 \varphi - (\varphi \partial) \varphi], \quad (2.3)$$

where D_F is the random force correlator (2.2) and the required integrations over $x = \{t, \mathbf{x}\}$ and summations over the vector indices are understood.

The standard RG technique can be applied to the model (2.3) if the function $d_F(k)$ is chosen in the form

$$d_F(k) = D_0 k^{1-2\varepsilon} h(m/k), \quad h(0) = 1. \quad (2.4)$$

Here D_0 is the positive amplitude factor, $m = 1/L$ is the reciprocal of the integral turbulence scale L , the function $h(m/k)$ provides the IR regularization and the exponent $\varepsilon > 0$ plays the part of the RG expansion parameter, similar to that played by $\varepsilon = 4 - d$ in Wilson's theory of critical phenomena [3]. The real (physical) value of this parameter is $\varepsilon = 2$: idealized energy injection by infinitely large eddies corresponds to $d_F(k) \propto \delta(\mathbf{k})$, and the function (2.4) for $\varepsilon \rightarrow 2$ and the appropriate choice of the amplitude can be considered as a power-law model of the three-dimensional δ function.

The model (2.3) is logarithmic (the coupling constant $g_0 \equiv D_0/\nu_0^3$ is dimensionless) at $\varepsilon = 0$, and the UV divergences have the form of the poles in ε in the correlation functions of the fields φ and φ' . Superficial UV divergences, whose removal requires counterterms, are present only in the 1-irreducible function $\langle \varphi' \varphi \rangle$, and the corresponding counterterm has the form $\varphi' \partial^2 \varphi$. Thus for the complete elimination of the UV divergences it is sufficient to perform the multiplicative renormalization of the parameters ν_0 and $g_0 = D_0/\nu_0^3$ with the only independent renormalization constant Z_ν :

$$\nu_0 = \nu Z_\nu, \quad g_0 = g \mu^{2\varepsilon} Z_g, \quad Z_g = Z_\nu^{-3} \quad (D_0 = g_0 \nu_0^3 = g \mu^{2\varepsilon} \nu^3). \quad (2.5)$$

Here μ is the reference mass in the minimal subtraction (MS) scheme, which we always use in what follows, g and ν are renormalized analogues of the bare parameters g_0 and ν_0 , and $Z = Z(g, \varepsilon, d)$ are the renormalization constants. In the MS scheme they have the form "1 + only poles in ε ," in particular,

$$Z_\nu = 1 + \sum_{k=1}^{\infty} a_k(g) \varepsilon^{-k} = 1 + \sum_{n=1}^{\infty} g^n \sum_{k=1}^n a_{nk} \varepsilon^{-k}, \quad (2.6)$$

with the one-loop coefficient $a_{11} = -1/40\pi^2$ [6,7].

III. TWO-LOOP APPROXIMATION FOR THE RG FUNCTIONS, FIXED POINT AND THE UV CORRECTION EXPONENT

We have performed calculation of the constant Z_ν with the accuracy of $O(g^2)$ (two-loop approximation). The calculation is rather involved and will be presented elsewhere (some details can be found in [10]), and below we give only the results for the residues a_{22} and a_{21} at the second-order and first-order poles in ε in the representation (2.6):

$$a_{22}/a_{11}^2 = 1, \quad a_{21}/a_{11}^2 \simeq -1.65. \quad (3.1)$$

The knowledge of the renormalization constant Z_ν to order $O(g^2)$ allows for the calculation of the RG functions, the anomalous dimension γ_ν and the beta function $\beta_g(\varepsilon, g)$, with the following accuracy:

$$\beta(g, \varepsilon) \equiv \tilde{\mathcal{D}}_\mu g = g(-2\varepsilon + 3\gamma_\nu(g)),$$

$$\gamma_\nu(g) \equiv \tilde{\mathcal{D}}_\mu \ln Z_\nu = -2g\partial_g a_1(g) = -2(a_{11}g + 2a_{21}g^2) + O(g^3), \quad (3.2)$$

where $\tilde{\mathcal{D}}_\mu$ is the operation $\mu\partial/\partial\mu$ at fixed bare parameters. In the MS scheme only the residues at the first-order poles in ε , that is, only the coefficients a_{k1} , contribute to the RG functions owing to the UV finiteness of the latter.

The coordinate of the fixed point is determined by the condition that $\beta(g_*) = 0$. From (3.1) and (3.2) we thus obtain:

$$g_* = (40\pi^2\varepsilon/3)(1 + \lambda\varepsilon) + O(\varepsilon^3), \quad \lambda \equiv 2a_{21}/3a_{11}^2 \simeq -1.10. \quad (3.3)$$

The correction exponent ω is determined by the slope of the beta function at the fixed point, $\omega = \beta'(g_*)$. Thus from (3.2) and (3.3) we obtain the first and second terms of its ε expansion:

$$\omega = 2\varepsilon(1 - \lambda\varepsilon) + O(\varepsilon^3). \quad (3.4)$$

IV. TWO-LOOP CALCULATION OF THE KOLMOGOROV CONSTANT

The Kolmogorov constant C_K can be defined as the (dimensionless) coefficient in the inertial-range expression $S_2(r) = C_K(\bar{\varepsilon}r)^{2/3}$ for the second-order structure function, predicted by the Kolmogorov–Obukhov theory and confirmed by experiment [1,2]. Here $\bar{\varepsilon}$ is the mean energy dissipation rate and the n -th order (longitudinal, equal-time) structure function is defined as

$$S_n(r) \equiv \langle [\varphi_r(t, \mathbf{x} + \mathbf{r}) - \varphi_r(t, \mathbf{x})]^n \rangle, \quad \varphi_r \equiv (\varphi_i \cdot r_i)/r, \quad r \equiv |\mathbf{r}|. \quad (4.1)$$

Using the exact relation $S_3(r) = -4\bar{\varepsilon}r/5$ that follows from the energy balance equation [1,2], the constant C_K can be related to the inertial-range skewness factor:

$$\mathcal{S} \equiv S_3/S_2^{3/2} = -(4/5)C_K^{-3/2}.$$

Many studies have been devoted to the derivation of C_K within the framework of the RG approach; see Refs. [11–18]. In order to obtain C_K , it is necessary to augment the solution of the RG equation for S_2 by some formula that relates the amplitude D_0 in the random force correlator (2.4) to the physical parameter $\bar{\varepsilon}$. In particular, in [11,12] the first-order term of the ε expansion for the pair correlator was combined with the so-called eddy damped quasi-Markovian approximation for the energy transfer function, taken at $\varepsilon = 2$. More elementary derivation, based on the exact relation between $\bar{\varepsilon}$ and the function $d_F(k)$ from (2.4) was given in [13]; see also [8,9]. In spite of the reasonable agreement with the experiment, such derivations are not immaculate from the theoretical viewpoints. Their common flaw is that the relation between $\bar{\varepsilon}$ and D_0 is unambiguous only in the limit $\varepsilon \rightarrow 2$:

$$\lim_{\varepsilon \rightarrow 2} \frac{D_0}{4\pi^2(2 - \varepsilon)} = \bar{\varepsilon}, \quad (4.2)$$

so that the coefficients of the corresponding ε expansions appear in fact arbitrary; see the discussion in [16] and Sec. 2.10 of [9]. The ambiguity is a consequence of the fact that the notion of the Kolmogorov constant has no definite extension to the nonphysical range $\varepsilon < 2$.

The experience on the RG theory of critical behaviour shows that unambiguous ε expansions can be written for universal quantities, such as critical exponents, normalized scaling functions and ratios of amplitudes in scaling laws [3]. The constant C_K extended to the range $\varepsilon < 2$ as in [11–13] involves a bare parameter, D_0 , and hence is not universal.

To circumvent this difficulty, we propose below an alternative derivation that relates C_K to an universal quantity and thus does not involve any relation between D_0 and $\bar{\varepsilon}$, and calculate C_K to second order of the expansion in ε (previous attempts have been confined with the first order). Consider the ratio

$$Q(\varepsilon) \equiv r\partial_r S_2(r)/|S_3(r)|^{2/3} = r\partial_r S_2(r)/(-S_3(r))^{2/3}. \quad (4.3)$$

The operation $r\partial_r \equiv r\partial/\partial r$ kills the constant contribution $\langle \varphi^2 \rangle$ in S_2 that diverges as $\Lambda \rightarrow \infty$ for $\varepsilon < 3/2$; in S_3 such constant contributions are absent.

Solving the RG equations for the quantities in $Q(\varepsilon)$ in the IR range ($\Lambda r \gg 1$) for general $0 < \varepsilon \leq 2$ gives

$$S_3(r) = D_0 r^{-3\Delta_\varphi} f_3(\varepsilon), \quad r\partial_r S_2(r) = D_0^{2/3} r^{-2\Delta_\varphi/3} f_2(\varepsilon), \quad \Delta_\varphi = 1 - 2\varepsilon/3; \quad (4.4)$$

see e.g. [8,9]. Thus the quantity $Q(\varepsilon) = f_2/(-f_3)^{2/3}$ in (4.3) does not depend on D_0 and can be calculated in the form of a regular ε expansion. We calculated the scaling functions $f_{2,3}$ to the second order of the ε expansion, which corresponds to the two-loop approximation in (3.3), (3.4), and obtained:

$$Q(\varepsilon) = (1/3)(20\varepsilon)^{1/3} [1 + 0.525\varepsilon + O(\varepsilon^2)]. \quad (4.5)$$

It is worth noting that the ε expansion for f_3 can be obtained not only from the direct perturbative calculation, but also from the exact expression

$$S_3(r) = -\frac{6\Gamma(2-\varepsilon)}{2^{2\varepsilon}\pi^{3/2}\Gamma(3/2+\varepsilon)} D_0 r^{-3\Delta_\varphi} \quad (4.6)$$

that follows from the energy balance equation and in the limit $\varepsilon \rightarrow 2$, along with the formula (4.2), reproduces the correct coefficient $-4/5$ (see above).

The value of $Q(\varepsilon)$ at $\varepsilon = 2$ determines the Kolmogorov constant and skewness factor through the exact relations $C_K = 6 \cdot 10^{-2/3} Q(2)$, $\mathcal{S} = -[1.5 \cdot Q(2)]^{-3/2}$, which follow from the definitions and the identity $r\partial_r r^\lambda = \lambda r^\lambda$ for any λ . Substituting the value of (4.5) we obtain $C_K = 3.02$, $\mathcal{S} = -0.15$. If we retained only the first-order term in (4.5) we would have obtained $C_K = 1.47$ and $\mathcal{S} = -0.45$. We also recall the experimental estimates recommended in [1]: $C_K \approx 1.9$ and $\mathcal{S} \approx -0.28$.

V. CONCLUSION

We have accomplished the complete two-loop calculation of the renormalization constant and RG functions for the stochastic problem (2.1)–(2.4) and derived the coordinate of the fixed point, the UV correction exponent ω , the Kolmogorov constant C_K and the inertial-range skewness factor \mathcal{S} to the second order of the corresponding ε expansions. The new point is not only the inclusion of the second-order correction, but also the derivation of C_K through an universal (in the sense of the theory of critical behaviour) quantity.

Of course, one should have not expected that the second-order terms of the ε expansions would be small in comparison to the first-order terms. The experience on the RG theory of critical behaviour shows that such corrections are not small for dynamical models (in contrast to static ones) and for amplitudes (in contrast to exponents); see [3]. It is thus rather surprising that in our case the account of the two-loop contributions leads to reasonable changes in the results.

Although the ε^2 correction to ω in (3.4) is rather large, it does not change its sign and hence does not destroy the IR stability of the fixed point.

The first-order approximation $C_K = 1.47$ underestimates, and the second-order approximation $C_K = 3.02$ overestimates the conventional experimental value of the Kolmogorov constant $C_K \approx 1.9$ [1]. Thus the experimental value of C_K (and hence for \mathcal{S}) lies in between of the two consecutive approximations. A similar situation is encountered for the well-known Heisenberg model [1], where the analogue of the Kolmogorov constant is known exactly and lies between the first-order and second-order approximations given by the corresponding ε expansion [19]. If we assume, by the analogy with the Heisenberg model, that the (unknown) exact predictions for C_K and \mathcal{S} lie between the first two approximations, we may conclude that our calculation has given a very good estimate for these quantities.

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